Statistical analysis of spatio-temporal and multi-dimensional data from a network of sensors

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- Spatio-temporal: Observations along time per sensor (node).
- 3 diverse forms: Observation per sensor (node) per time is scalar/vector/distribution.

Data illustration: scalar observation

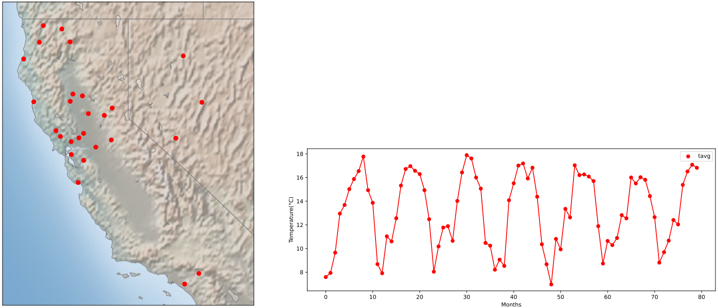

Figure 1: *Monthly climatological records of weather stations in California.* A value $x_{it} \in \mathbb{R}$ is recorded on each station (sensor/node) i, at each time t. In this example, x_{it} is the average temperature.

Data illustration: vectorial observation

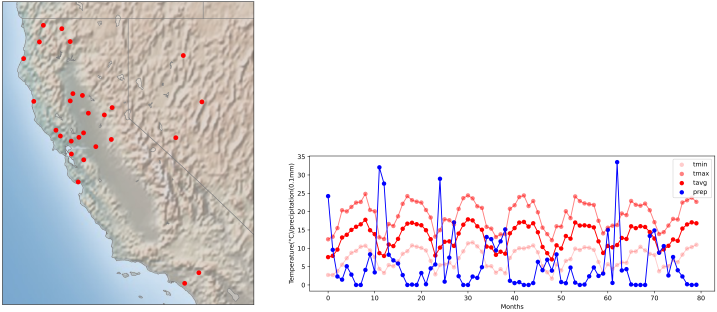

Figure 2: *Monthly climatological records of weather stations in California.* A vector $\mathbf{x}_{it} \in \mathbb{R}^4$ is recorded on each station (sensor/node) i, at each time t. In this example, $\mathbf{x}_{it}$ is the vector of: min/max/avg temperature, and precipitation.

Data illustration: distributional observation

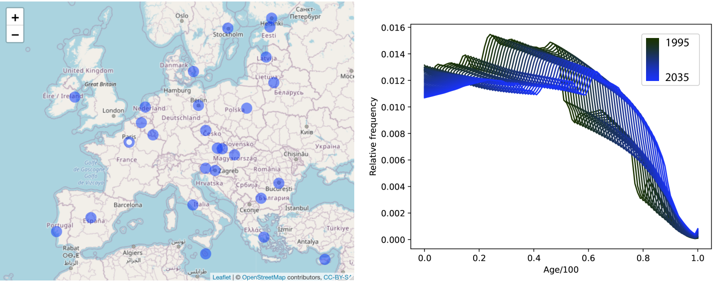

Figure 3: *Annual records of age distributions of EU countries.* A distribution $\boldsymbol{\mu}_{it} \in \mathcal{P}([0,1])$ is recorded on each node i, at each time t. In this example, $\boldsymbol{\mu}_{it}$ is an age distribution. Time is represented by color instead of x-axis. Lighter curves correspond to the distributions from more recent years.

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Causal graph and vector auto-regressive model

For $(x_{it})_t \in \mathbb{R}$, $i = 1, \ldots, N$, the **VAR Models** have been widely adapted in literature to learn their causality (Granger) dependency.

$$\text{VAR}(1): \; \mathbf{x}_t - \mathsf{u} = A(\mathbf{x}_{t-1} - \mathsf{u}) + \mathbf{z}_t, \tag{1}$$

where $\mathbf{x}_t = (x_{1t}, \ldots, x_{Nt})$, $\mathsf{u} = \mathbb{E}\mathbf{x}_t$, and $\mathbf{z}_t$ is white noise.

When VAR (1) is stationary, the sparsity structure of $A \overset{\text{adj. mat.}}{\Longleftrightarrow} \mathcal{G}$.

Contribution of the thesis:

$$\mathcal{G} \text{ of } (x_{it})_t \in \mathbb{R} \to \begin{cases} \mathcal{G} \text{ of } (\mathbf{x}_{it})_t \in \mathbb{R}^F + \text{online inference}, \\ \mathcal{G} \text{ of } (\boldsymbol{\mu}_{it})_t \in \mathcal{W}_2(\mathbb{R}). \end{cases}$$

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Kronecker sum, causal product graph and Matrix AR

$$(\mathbf{x}_{it})_t \in \mathbb{R}^F,\ i = 1, \dots, N \iff (\mathbf{X}_t)_t \in \mathbb{R}^{N \times F},$$

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where $\mathbf{x}_t = \mathrm{vec}(\mathbf{X}_t) = (\boldsymbol{x}_{ift})_{i,f}$, $A_{\mathrm{N}} \in \mathbb{R}^{N \times N}$, $A_{\mathrm{F}} \in \mathbb{R}^{F \times F}$, and

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KS endows the matrix representation of the vector Model $(1')$:

$$\mathbf{X}_t - \mathsf{U} = A_{\mathrm{N}}(\mathbf{X}_{t-1} - \mathsf{U}) + (\mathbf{X}_{t-1} - \mathsf{U})A_{\mathrm{F}}^\top + \mathbf{Z}_t,$$

$A_{\mathrm{N}} \sim$ spatial dependency, $A_{\mathrm{F}} \sim$ feature dependency.

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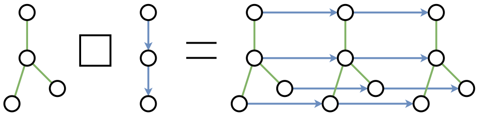
$$\mathcal{G} = \mathcal{G}_N \square \mathcal{G}_F, \text{ where } \mathcal{G}_N \overset{\text{adj. mat.}}{\Longleftrightarrow} A_\mathrm{N} \text{ and } \mathcal{G}_F \overset{\text{adj. mat.}}{\Longleftrightarrow} A_\mathrm{F}.$$



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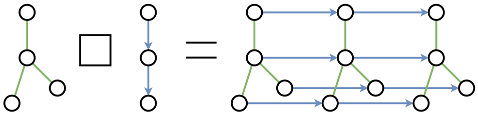

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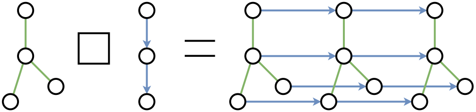

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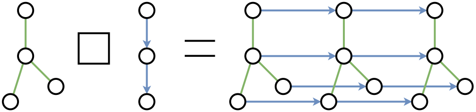

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$\mathcal{G}_N$ = spatial graph of $(\mathbf{x}_{it})_t$, $\mathcal{G}_F$ = feature graph.

Constraint set $\mathcal{K}_{\mathcal{G}}$

Due to the model identifiability and application reasons, we employ a more sophisticated structure for A. The complete MAR(1) is

$$\mathbf{x}_t = A\mathbf{x}_{t-1} + \mathbf{z}_t, \; A \in \mathcal{K}_{\mathcal{G}},$$

where $\mathbf{x}_t = \text{vec}\left(\mathbf{X}_t\right)$, and

$$\begin{aligned} \mathcal{K}_{\mathcal{G}} = & \{\mathsf{M} \in \mathbb{R}^{NF \times NF} : \exists\, \mathsf{M}_{\mathrm{F}} \in \mathbb{R}^{F \times F}, \mathsf{M}_{\mathrm{N}} \in \mathbb{R}^{N \times N}, \text{ such that,} \\ & \text{offd}(\mathsf{M}) = \mathsf{M}_{\mathrm{F}} \oplus \mathsf{M}_{\mathrm{N}}, \text{ with, } \text{diag}(\mathsf{M}_{\mathrm{F}}) = 0, \; \text{diag}(\mathsf{M}_{\mathrm{N}}) = 0, \\ & \mathsf{M}_{\mathrm{F}} = \mathsf{M}_{\mathrm{F}}^{\top}, \; \mathsf{M}_{\mathrm{N}} = \mathsf{M}_{\mathrm{N}}^{\top}\}, \end{aligned}$$

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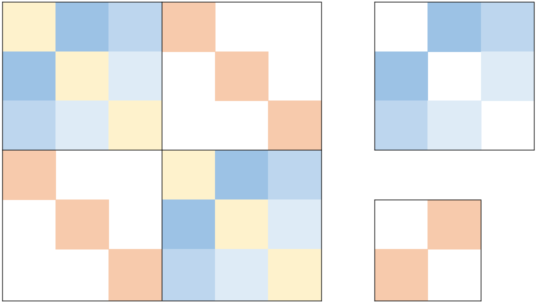

Figure 4: $\mathcal{K}_{\mathcal{G}}$ *for* $N = 3$, $F = 2$. M (left), M_{N} (right upper), M_{F} (right bottom).

Matrix AR(1)

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$\mathbf{z}_t \in \mathbb{R}^{NF} \sim \mathrm{IID}\,(0, \Sigma)$ is white noise with a non-singular covariance structure Σ and bounded fourth moments, with $\|A\|_2 < 1$.

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$\lambda_t \to \lambda_{t+1}$, $\mathbf{A}(t,\lambda_t) \to \mathbf{A}(t,\lambda_{t+1})$, $\mathbf{A}(t,\lambda_{t+1}) \to \mathbf{A}(t+1,\lambda_{t+1})$.

Homotopy algorithms and optimality conditions

$$\boldsymbol{\theta}^* = \arg\min_{\theta \in \mathbb{R}^d} L(\theta),\ L(\theta) = \frac{1}{2t}\|\mathbf{y} - \mathbf{X}\theta\|_{\ell_2}^2 + \lambda\|\theta\|_{\ell_1},$$

Algo: $\boldsymbol{\theta}^*(\lambda_1) \rightarrow \boldsymbol{\theta}^*(\lambda_2)$ relies on **Optimality condition** of minimizer $\boldsymbol{\theta}^*$:

$$\frac{\partial L(\theta)}{\partial \theta} = 0 \iff \mathbf{X}^\top(\mathbf{X}\boldsymbol{\theta}^* - \mathbf{y}) + \lambda\mathbf{w} = 0,\ \mathbf{w} = \partial\|\boldsymbol{\theta}^*\|_{\ell_1}.$$

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Unique $\boldsymbol{\theta}^* = (\boldsymbol{\theta}_1^*, 0)$ at λ, $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_0)$, $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_0)$:

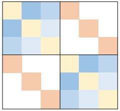
$$\begin{cases} \boldsymbol{\theta}_1^* = (\mathbf{X}_1^\top \mathbf{X}_1)^{-1}(\mathbf{X}_1^\top \mathbf{y} - \lambda \mathbf{w}_1), \\ \lambda \mathbf{w}_0 = \mathbf{y} - \mathbf{X}_0^\top \mathbf{X}_1 \boldsymbol{\theta}_1^*. \end{cases}$$

Continuity $\implies$ (8) is the explicit form of all lasso solutions in a neighbourhood of λ, which ends with the critical values.

Sub-gradients under the structure constraint

$$\min_{A \in \mathcal{K}_{\mathcal{G}}} \frac{1}{2t} \sum_{\tau=1}^{t} \|\mathbf{x}_\tau - A\mathbf{x}_{\tau-1}\|_{\ell_2}^2 + \lambda F \|A_{\mathrm{N}}\|_{\ell_1}$$

$? \dfrac{\partial L(A)}{\partial A}$ while $A \in \mathcal{K}_{\mathcal{G}}$



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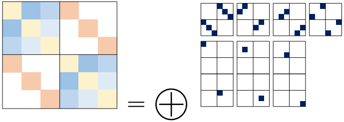


$$\mathcal{K}_{\mathcal{G}} = \bigoplus_{k \in K} \text{span}\{\widetilde{U}_k\} \Rightarrow A = \sum_{k \in K} \langle \widetilde{U}_k, A^0 \rangle_F \widetilde{U}_k, \text{ where}$$

$$I_F \otimes A_{\mathrm{N}} = \sum_{k \in K_{\mathrm{N}}} \langle \widetilde{U}_k, A^0 \rangle_F \widetilde{U}_k, K_{\mathrm{N}} \subset K.$$

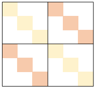
Lasso above becomes

$$\min_{A^0 \in \mathbb{R}^{NF \times NF}} \frac{1}{2t} \sum_{\tau=1}^{t} \left\| \mathbf{x}_\tau - \sum_{k \in K} \langle U_k, A^0 \rangle U_k \mathbf{x}_{\tau-1} \right\|_{\ell_2}^2 + \lambda \left\| \sum_{k \in K_{\mathrm{N}}} \langle U_k, A^0 \rangle U_k \right\|_{\ell_1}$$

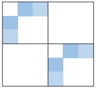
Sub-gradients under the structure constraint

$\frac{\partial L(A^0)}{\partial A^0} = 0 \Longrightarrow$ The optimality condition of $\boldsymbol{A} \in \mathcal{K}_{\mathcal{G}}$:

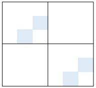
$$\text{Proj}_{\text{DF}}\left(\mathbf{A}\widehat{\mathbf{\Gamma}}_t(0) - \widehat{\mathbf{\Gamma}}_t(1)\right) = 0,$$



$$\text{Proj}_{K_{\text{N}}^1}\left(\mathbf{A}\widehat{\mathbf{\Gamma}}_t(0) - \widehat{\mathbf{\Gamma}}_t(1)\right) + \lambda I_F \otimes \boldsymbol{W}^1 = 0,$$



$$\text{Proj}_{K_{\text{N}}^0}\left(\mathbf{A}\widehat{\mathbf{\Gamma}}_t(0) - \widehat{\mathbf{\Gamma}}_t(1)\right) + \lambda I_F \otimes \boldsymbol{W}^0 = 0,$$



where $\widehat{\mathbf{\Gamma}}_t(0) = \sum_{\tau=1}^{t} \mathbf{x}_{\tau-1}\mathbf{x}_{\tau-1}^\top, \widehat{\mathbf{\Gamma}}_t(1) = \sum_{\tau=1}^{t} \mathbf{x}_{\tau}\mathbf{x}_{\tau-1}^\top$, $\boldsymbol{W}^0$ is the sub-gradient matrix of zero entries in $\boldsymbol{A}_{\mathbf{N}}$, and $\boldsymbol{W}^1$ is the sign matrix of active entries in $\boldsymbol{A}_{\mathbf{N}}$.

Adaptive tuning of lambda

$$\mathbf{A}(t, \lambda_t) \rightarrow \mathbf{A}(t, \lambda_{t+1}), \mathbf{A}(t, \lambda_{t+1}) \rightarrow \mathbf{A}(t+1, \lambda_{t+1})$$

$\lambda_t \rightarrow \lambda_{t+1}$:
Monti et al. (2018); Garrigues and Ghaoui (2008) propose an adaptive tuning method, in our notations:

$$f_{t+1}(\lambda) = \frac{1}{2}\|\mathbf{x}_{t+1} - \mathbf{A}(t, \lambda)\mathbf{x}_t\|_{\ell_2}^2,$$

and updating rule:

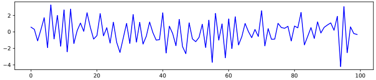
$$\lambda_{t+1} = \lambda_t - \eta \frac{\mathrm{d} f_{t+1}(\lambda)}{\mathrm{d}\lambda}|_{\lambda=\lambda_t},$$

$\frac{\mathrm{d}\mathbf{A}(t,\lambda)}{\mathrm{d}\lambda}|_{\lambda=\lambda_t}$ can be calculated from the optimality condition of $\mathbf{A}(t, \lambda_t)$.

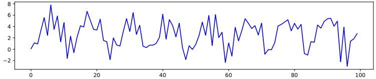
Online graph and trend learning

Come back to the assumption:

$\mathbb{E}(\mathbf{x}_\tau)_\tau = 0, \ \forall \tau \Rightarrow \frac{1}{t}\sum_{\tau=1}^{t} \|\mathbf{x}_\tau - A\mathbf{x}_{\tau-1}\|_{\ell_2}^2 .$



However, raw data $\mathbb{E}(\mathbf{x}_\tau)_\tau = \mathrm{b}_\tau$, that is, a trend is present.



Offline: Detrend $\mathbf{x}_\tau - \widehat{\mathrm{b}}_\tau \Rightarrow$ is forbidden online.

Online graph and trend learning

Augmented data model:

$$\begin{cases} \mathbf{x}_t = \mathsf{b}_t + \mathbf{x}'_t, \rightarrow \text{ Observations} \\ \mathbf{x}'_t = \mathbf{A}\mathbf{x}'_{t-1} + \mathbf{z}_t, \rightarrow \text{ underlying stationary process.} \end{cases}$$

In particular, we consider periodic trend of period M:

$$\mathbf{b}_t = \mathbf{b}_m,\ m = 0, ..., M-1,\ m = t \bmod M.$$

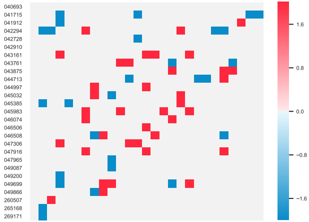
Augmented structured matrix Lasso:

$$\underset{A \in \mathcal{K}_{\mathcal{G}}, \mathsf{b}_m}{\arg\min} \frac{1}{2t} \sum_{m=0}^{M-1} \sum_{\tau \in I_{m,t}} \|(\mathbf{x}_\tau - \mathsf{b}_m) - A(\mathbf{x}_{\tau-1} - \mathsf{b}_{m-1})\|_{\ell_2}^2 + \lambda_t F \|A_{\mathrm{N}}\|_{\ell_1},$$

where $I_{m,t} = \{\tau = 1, ..., t : \tau \bmod M = m\}$.

Detrend + graph estimation simultaneously
$\Longrightarrow$ Online graph learning on raw data

Climatology data



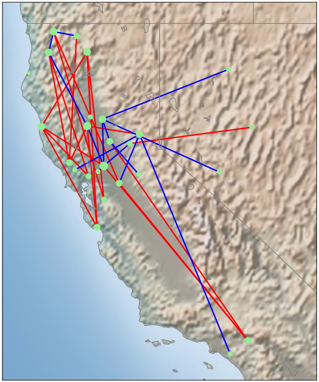
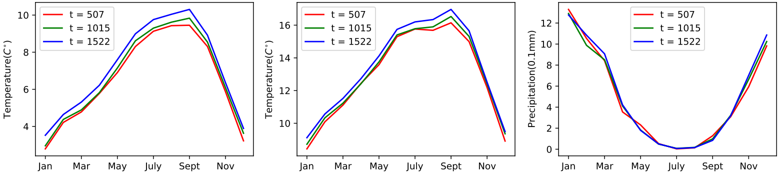


Figure 5: *California weather graph.* Graph Adjacency matrix (left), visualization on the map (right) using sensor coordinates. The nodes with bigger sizes connect with more nodes.

Climatology data



Minimal temperature Average temperature Precipitation

Figure 6: *Estimated trends along years.* On the left, middle, right are the estimated trends at different years of a certain station for the 3 features. Experiment settings: $N = 27$, $F = 4$, $M = 12$.

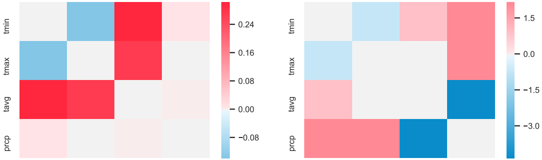
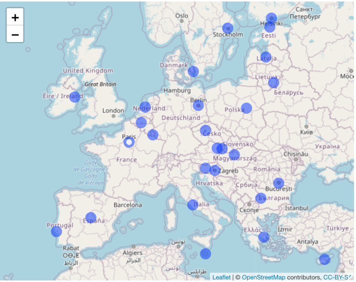


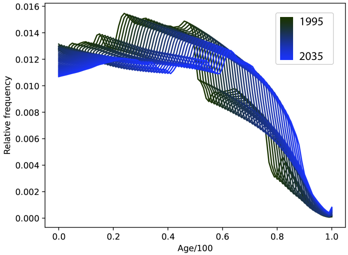
Figure 7: *Updated feature graph at $t = 1522$.* Projected OLS (left), and Lasso (right). Experiment settings: $N = 27$, $F = 4$, $M = 12$.

Note that $t = 1522 < \#params = 1761$.

Table of Contents

Recall the data setting:





Learn $\mathcal{G}$ of $(\boldsymbol{\mu}_{it})_t \in \mathcal{W}_2(\mathbb{R}), i = 1, \ldots N$ with a multivariate distributional AR model.

Random probability measures in Wasserstein space

$$\mathcal{W}_2(\mathbb{R}) = \left\{ \mu \in \mathcal{P}(\mathbb{R}) \middle| \int_{\mathbb{R}} x^2 d\mu(x) < \infty \right\},$$

endowed with the 2-Wasserstein distance

$$d_W(\mu, \gamma)^2 = \inf_{\pi \in \Pi(\mu,\gamma)} \int_{\mathbb{R} \times \mathbb{R}} (x_1 - x_2)^2 d\pi(x_1, x_2)$$

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$$\overset{\|\cdot\|^2 + \mathbb{R}}{=} \int_0^1 \left(F_\mu^{-1}(u) - F_\gamma^{-1}(u) \right)^2 du,$$

where $F_\mu^{-1}(u), F_\gamma^{-1}(u)$ are the quantile functions of μ and γ.

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where $F_\mu^{-1}(u), F_\gamma^{-1}(u)$ are the quantile functions of μ and γ.

$\mathcal{W}_2$ *is not linear space.* Chen et al. (2021); Zhang et al. (2021); Zhu and Müller (2021) extended the univariate AR model

$$\boldsymbol{x}_t - u = \alpha(\boldsymbol{x}_{t-1} - u) + \boldsymbol{\epsilon}_t,$$

by relying on the notion of *Tangent space* in $\mathcal{W}_2$.

Enable again linear methods - Tangent space

$\mathcal{W}_2 := \mathcal{W}_2(\mathbb{R})$ has a pseudo-Riemannian structure (Ambrosio et al., 2008).

Let $\gamma \in \mathcal{W}_2$ be an atomless measure (that is it possesses a continuous cdf F_γ), the tangent space at γ is defined as

$$\mathrm{Tan}_\gamma = \overline{\{t(T_\gamma^\mu - id) : \mu \in \mathcal{W}_2,\ t > 0\}}^{\mathcal{L}^2_\gamma},$$

where $T_\gamma^\mu = F_\mu^{-1} \circ F_\gamma$ is the optimal map, that pushes γ forward to μ.

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$$\langle f, g \rangle_\gamma := \int_{\mathbb{R}} f(x) g(x)\, d\gamma(x),\ f, g \in \mathcal{L}^2_\gamma(\mathbb{R}),$$

and the induced norm $\| \cdot \|_\gamma$.

Enable again linear methods - Tangent space

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Definition

The logarithmic map $\mathrm{Log}_\gamma : \mathcal{W}_2 \to \mathrm{Tan}_\gamma$ is defined as

$$\mathrm{Log}_\gamma \mu = T_\gamma^\mu - id.$$

The exponential map $\mathrm{Exp}_\gamma : \mathrm{Tan}_\gamma \to \mathcal{W}_2$ is defined as

$$\mathrm{Exp}_\gamma g = (g + id)\#\gamma,$$

where $T\#\mu$ is the measure pushforwarded by function T, defined as $[T\#\mu](A) = \mu(\{x : T(x) \in A\})$.

Line segment in Tangent space = geodesic in $\mathcal{W}_2$

The geodesic (McCann's interpolant) between γ and μ

$$\mathrm{Exp}_\gamma[\alpha(T_\gamma^\mu - id)], \; \alpha : 0 \to 1,$$

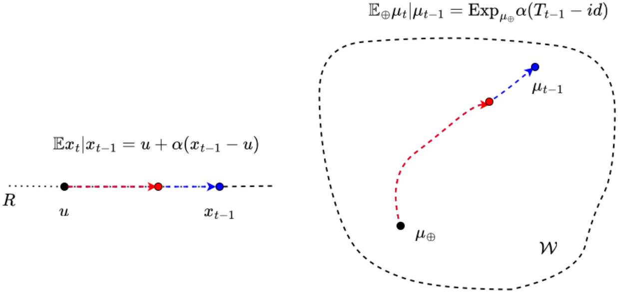
Line segment in Tangent space = geodesic in $\mathcal{W}_2$

The geodesic (McCann's interpolant) between γ and μ

$$
\begin{aligned}
&\mathrm{Exp}_\gamma[\alpha(T_\gamma^\mu - id)],\ \alpha : 0 \to 1, \\
&= [\alpha(T_\gamma^\mu - id) + id]\#\gamma
\end{aligned}
$$

Related work: Univariate Wasserstein AR model

Chen et al. (2021); Zhang et al. (2021); Zhu and Müller (2021) proposed to interpret the regression operation geometrically.



Let $\boldsymbol{\mu}$ be a random measure from $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathcal{W}_2$

$$\text{(Fréchet mean)} \quad \mathbb{E}_{\oplus}\boldsymbol{\mu} = \arg\min_{\nu \in \mathcal{W}_2} \mathbb{E}\left[d_W^2(\boldsymbol{\mu}, \nu)\right].$$

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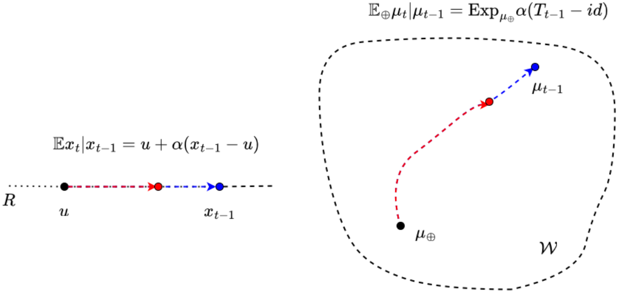

Let $\boldsymbol{\mu}$ be a random measure from $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathcal{W}_2$

(conditional Fréchet mean) $\mathbb{E}_{\oplus}\boldsymbol{\mu}|\boldsymbol{\gamma} = \underset{\nu \in \mathcal{W}_2}{\arg\min}\, \mathbb{E}\left[d_W^2(\boldsymbol{\mu}, \nu)|\boldsymbol{\gamma}\right].$

Multivariate Wasserstein AR model

Multivariate regression operation (VAR(1)):
for any fixed $i = 1, \dots N$

$$\mathbb{E}\boldsymbol{x}_{it}|\boldsymbol{x}_{j,t-1} = u_i + \sum_{j=1}^{N} A_{ij}(\boldsymbol{x}_{j,t-1} - u_j) \Rightarrow \begin{cases} \boldsymbol{T}_{1,t-1} - id & \in \mathrm{Tan}_{\mu_{1,\oplus}} \\ \boldsymbol{T}_{2,t-1} - id & \in \mathrm{Tan}_{\mu_{2,\oplus}} \\ & \vdots \end{cases}$$

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$\iff$

$$\begin{cases} \text{Center} & \tilde{\boldsymbol{x}}_{it} = \boldsymbol{x}_{it} - u_i, \quad \xrightarrow{\text{ref pt}} \mathbb{E}\tilde{\boldsymbol{x}}_{it} = 0, \\ \text{Push} & \mathbb{E}\tilde{\boldsymbol{x}}_{it}|\tilde{\boldsymbol{x}}_{j,t-1} = 0 + \sum_{j=1}^{N} A_{ij}\tilde{\boldsymbol{x}}_{jt}, \end{cases}$$

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$$\Longrightarrow \begin{cases} \text{Center} & \tilde{\boldsymbol{\mu}}_{it} = ? \xrightarrow{\text{ref pt}} \mathbb{E}_{\oplus}\tilde{\boldsymbol{\mu}}_{it} = c \\ \text{Push} & \mathbb{E}_{\oplus}\tilde{\boldsymbol{\mu}}_{it}|\tilde{\boldsymbol{\mu}}_{j,t-1} = \mathrm{Exp}_c\left(\sum_{j=1}^{N} A_{ij}(\tilde{\boldsymbol{T}}_{j,t-1} - id)\right) \end{cases}$$

Center a random measure $\boldsymbol{\mu}$, s.t. $\mathbb{E}_{\oplus}\boldsymbol{\mu} = U(0,1)$

Zhu and Müller (2021) proposed a notion of addition for two increasing functions:

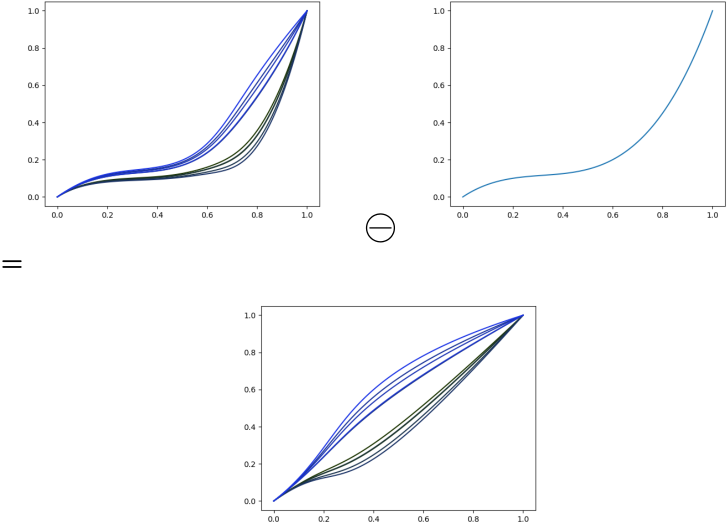
$$g \oplus f := g \circ f \implies g \ominus f := g \circ f^{-1},$$

where $^{-1}$ are the left continuous inverse.
For $\boldsymbol{\mu}$, its centered measure $\widetilde{\boldsymbol{\mu}}$ is defined by the quantile function

$$\widetilde{\boldsymbol{F}}_{\mu}^{-1} = \boldsymbol{F}_{\mu}^{-1} \ominus F_{\oplus}^{-1},$$

where $\boldsymbol{F}_{\mu}^{-1}, F_{\oplus}^{-1}$ et $\widetilde{\boldsymbol{F}}_{\mu}^{-1}$ are respectively quantile functions of $\boldsymbol{\mu}, \mu_{\oplus}$, and $\widetilde{\boldsymbol{\mu}}$.

Center a random measure $\boldsymbol{\mu}$, s.t. $\mathbb{E}_{\oplus}\boldsymbol{\mu} = U(0,1)$



Wasserstein multivariate AR Model

$$\widetilde{\boldsymbol{\mu}}_{it} = \boldsymbol{\epsilon}_{it} \# \operatorname{Exp}_{Leb}\left(\sum_{j=1}^{N} A_{ij}(\widetilde{\boldsymbol{F}}_{j,t-1} - id)\right),$$

where $\{\boldsymbol{\epsilon}_{it}\}_{i,t}$ are i.i.d. random increasing functions, $\boldsymbol{\epsilon}_{it}$ is almost surely independent of $\boldsymbol{\mu}_{j,t-1}$, $i, j = 1, \ldots, N$, for all $t \in \mathbb{Z}$, and

$$\mathbb{E}\left[\boldsymbol{\epsilon}_{it}(x)\right] = x,\ x \in [0,1].$$

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Assumption

- *All* $\boldsymbol{\mu}_{it}$, $t \in \mathbb{Z}$, $i = 1, \ldots, N$ *are supported on* $[0, 1]$.
- *(*N*-**simplex**)* $\sum_{j=1}^{N} A_{ij} \leqslant 1$ *and* $0 \leqslant A_{ij} \leqslant 1$.

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Quantile function representation

$$\widetilde{\boldsymbol{F}}_{i,t}^{-1} = \boldsymbol{\epsilon}_{i,t} \circ \left[\sum_{j=1}^{N} A_{ij} \left(\widetilde{\boldsymbol{F}}_{j,t-1}^{-1} - id \right) + id \right],$$

Wasserstein multivariate AR Model

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Existence, uniqueness, and stationarity

$$\widetilde{\boldsymbol{F}}_{i,t}^{-1} = \boldsymbol{\epsilon}_{i,t} \circ \left[\sum_{j=1}^{N} A_{ij} \left(\widetilde{\boldsymbol{F}}_{j,t-1}^{-1} - id \right) + id \right] \tag{8}$$

Admissible as a time series model: existence, uniqueness and stationarity of series $(\widetilde{\boldsymbol{F}}_{i,t}^{-1})_t, i = 1, \ldots N$.

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Admissible as a time series model: existence, uniqueness and stationarity of series $(\widetilde{\boldsymbol{F}}_{i,t}^{-1})_t, i = 1, \dots N$.

Theoretical results:

Under two classical conditions, we have proved:

• Iterated random function system (8) admits uniquely one solution in the metric space

$$(\mathcal{T}, \| \cdot \|_{Leb})^{\otimes N}, \mathcal{T} = \{F_\mu^{-1} | \mu \in \mathcal{W}_2(\mathbb{R})\}$$

• The unique solution is stationary (2nd order) in the Hilbert space

$$(\mathcal{T}, \langle , \rangle_{Leb})^{\otimes N}, \mathcal{T} = \{F_\mu^{-1} | \mu \in \mathcal{W}_2(\mathbb{R})\}$$

according to a proper definition for functional TS.

Constrained least-square estimation

For the auto-regressive model

$$\widetilde{\boldsymbol{F}}_{i,t}^{-1} = \boldsymbol{\epsilon}_{i,t} \circ \left[\sum_{j=1}^{N} A_{ij} \left(\widetilde{\boldsymbol{F}}_{j,t-1}^{-1} - id \right) + id \right],$$

given the centered observations $\widetilde{\boldsymbol{F}}_t^{-1}$, $t = 0, 1, \ldots, T$, we propose

$$\widetilde{\boldsymbol{A}}_{i:} = \underset{A_{i:} \in B_+^1}{\arg\min} \frac{1}{T} \sum_{t=1}^{T} \left\| \widetilde{\boldsymbol{F}}_{i,t}^{-1} - \sum_{j=1}^{N} A_{ij} \left(\widetilde{\boldsymbol{F}}_{j,t-1}^{-1} - id \right) - id \right\|_{Leb}^{2},$$

where B_+^1 is the constraint set of N-simplex.

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given the centered observations $\widehat{\boldsymbol{F}}_t^{-1}$, $t = 0, 1, \ldots, T$, we propose

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where B_+^1 is the constraint set of N-simplex.

Theoretical result:

$$\widehat{\boldsymbol{A}} \xrightarrow{p} A, \text{ as } T \to +\infty.$$

Age distribution of countries

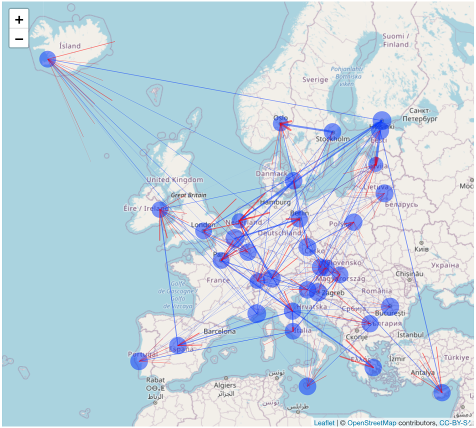

Figure 8: *Inferred age structure graph.* The non-zero coefficients A_{ij} are represented by the weighted directed edges from node j to node i. Thicker arrow corresponds to larger weights. The blue circles around nodes represent the weights of self-loop.

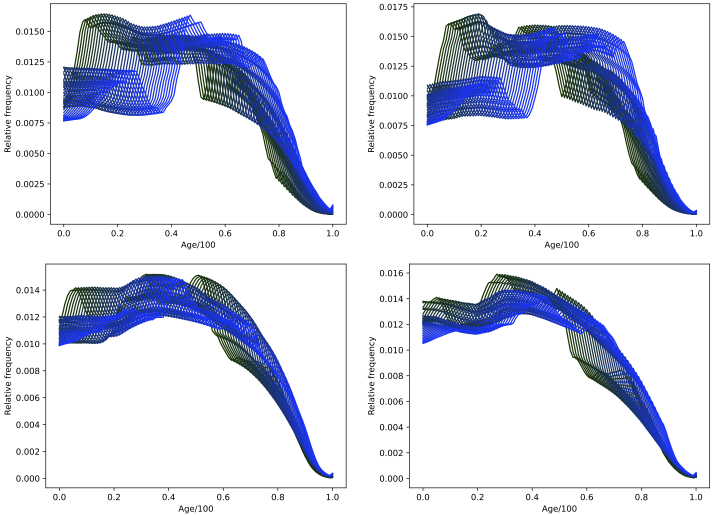

Figure 9: *Evolution of age structure from* 1995 *to* 2035 *(projected). Estonia (top left), Latvia(top right), Sweden (bottom left) versus Norway (bottom right).*

Age distribution of countries

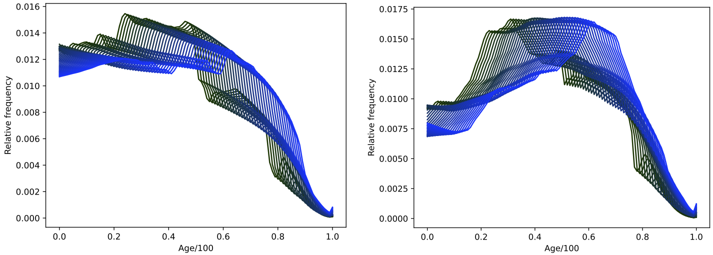

Figure 10: *Evolution of age structure from* 1995 *to* 2035 *(projected) of France (left) versus Italy (right).*

Goal: given $(\mathbf{x}_{nt})_t \in \mathbb{R},\ n \in \mathcal{N} = \{1, \ldots, N\}$, finding the highly predictable series $i \in I \subset \mathcal{N}$, such that their observations $\boldsymbol{x}_{it}$ can be reconstructed accurately by the past and present obs of other series $\boldsymbol{x}_{j\tau},\ j \in I^c, t - H \leqslant \tau \leqslant t$ in real time.

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$\mathcal{G}$ can be given, e.g. geographical graph of the weather stations.

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$\mathcal{G}$ can be given, e.g. geographical graph of the weather stations.

We use 3 prediction methods to evaluate the node predictability: *kernel ridge regression, linear regression, and neural networks*.

$\longrightarrow$ 3 ranking procedures.

In this thesis, we provided new statistical tools for analyzing spatio-temporal and multi-dimensional data. In particular, we extended the classical VAR(1) model for the complex data types: matrix-variate and distributional data in the way to serve graph learning.

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Future works:

These two works introduce a more general topic: **object data analysis**. Especially, the 2nd work demonstrates one important way to perform the analysis, that is to view data points as **random objects in a metric space**.

Graph itself is also an important data object. Among others, it is adopted to represent the brain functional connectivity.

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Already available graph-valued models:

- Network regression with Euclidean predictors (Zhou and Müller, 2021): $\mathbb{E}_{\oplus}\boldsymbol{G}|\mathbf{x}$. The model is applied to study the evolution of brain connectivity wrt age
- Two sample tests (Ginestet et al., 2017): $\boldsymbol{G}_i \overset{i.i.d.}{\sim} \boldsymbol{G}_1$, $\boldsymbol{G}_j \overset{i.i.d.}{\sim} \boldsymbol{G}_2 \to \mathbb{E}_{\oplus}\boldsymbol{G}_1 ? = \mathbb{E}_{\oplus}\boldsymbol{G}_2$. The model is applied to study the impact of gender on brain connectivity.

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More generalized models to be developed, e.g. Network (functional) regression with network predictors.

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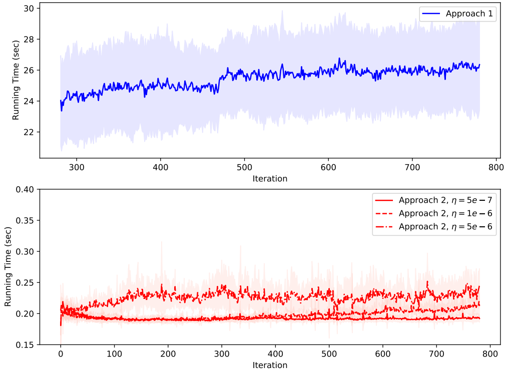

Figure 11: *Running time of each online update.* The red curves are the mean running time of the high-dimensional procedure, taken over 10 simulations each. The blue curve is the mean running time of the low-dimensional procedure, taken over the same 30 simulations. The shaded areas represent the corresponding one standard deviations. Other simulation settings: $N = 20$, $F = 5$, $M = 12$, number of model parameters $= 1500$.

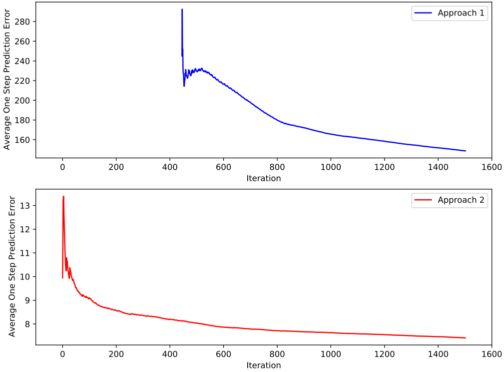

Figure 12: *Average one step prediction error of raw time series.* Projected OLS (top), and Lasso (bottom).

Homotopy algorithms and optimality conditions

Algo: $t_1 \to t_2$:

$$\theta^* = \arg\min_{\theta \in \mathbb{R}^d} L(\theta),\ L(\theta) = \frac{1}{2} \left\| \begin{pmatrix} \mathbf{y} \\ \mu y_{t+1} \end{pmatrix} - \begin{pmatrix} \mathbf{X} \\ \mu \mathbf{x}_{t+1}^\top \end{pmatrix} \theta \right\|_{\ell_2}^2 + \lambda \|\theta\|_{\ell_1},$$

$\frac{\partial L}{\partial \theta} = 0 \Longrightarrow$ Optimality condition $\Longrightarrow$ Homotopy algorithm.

Enable again linear methods - towards tangent space

$$d_W(\mu,\gamma)^2 = \inf_{\pi\in\Pi(\mu,\gamma)} \int_{\mathbb{R}\times\mathbb{R}} (x_1 - x_2)^2 d\pi(x_1, x_2)$$

When γ is an atomless measure, that is F_γ is continue, we have π^* exists uniquely and is induced by a function $T_\gamma^\mu : \mathbb{R} \to \mathbb{R}$, such that

$$T_\gamma^\mu \# \gamma = \mu$$

where $[T_\gamma^\mu \# \gamma](A) = \gamma(\{x : T_\gamma^\mu(x) \in A\}),\ A \subset R$. T_γ^μ is called optimal transport map. Furthermore,

$$T_\gamma^\mu(x) = F_\mu^{-1} \circ F_\gamma(x).$$

Characterization of the difference between μ, γ:

$$d_W(\mu,\gamma) \Longrightarrow T_\gamma^\mu(x) \in \mathcal{L}_\gamma^2(\mathbb{R}).$$

Adaptive tuning of lambda

Updating rule can be interpreted as the steps in the **projected stochastic gradient descent** derived for the batch problem

$$\lambda_n^* = \arg\min_{\lambda \geqslant 0} \frac{1}{2n} \sum_{t=1}^{n} \|\mathbf{x}_{t+1} - \mathbf{A}(t,\lambda)\mathbf{x}_t\|_{\ell_2}^2,$$

which is the average one step prediction error.

KS/KP as a common practice

vec$(\mathbf{X}_i)+$ **vector model + KP/KS imposed in parameters** is a common practice to extend vector models to matrix-variate data in literature. For example, Gupta and Nagar (2018) proposed a matrix-variate Normal distribution as:

$$\text{vec}(\mathbf{X}_i) \overset{iid}{\sim} \mathcal{N}(\text{u}, \Sigma), \text{ where } \Sigma = \Sigma_1 \otimes \Sigma_2,$$

where $\otimes$ is the Kronecker product.